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Classical Limit of a Relativistic Quantum System

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The Poincaré-covariant Hamiltonian particle dynamics proposed by Aaberge is formulated in terms of W^* algebras in the standard representation. In this way a unified description of both quantum and classical systems is provided. The classical limit ($\hbar \to 0$) of a quantum system is obtained. The general theory thus formulated includes systems with superselection rules in a natural way.

1. INTRODUCTION

A theory of relativistic, interacting particles, classical as well as quantal, has been proposed by Aaberge (1977, 1978). Application to systems in two different areas, namely, quantal Coulomb systems (Aaberge, to be published) and classical gravitational systems (Aaberge, 1979), provides some empirical evidence for the theory. We assume it to be the correct theory for particles with constant internal properties, interacting via classical fields.

With this theory to our disposition, it becomes possible to study the classical limit of a relativistic quantum system of the kind described above in a rigorous way. Of related, but greater, importance with regard to applications, is the fact that it becomes possible to study systems with both quantal and classical properties, i.e., systems with superselection rules. The necessity of this in the area of molecular theory, has been pointed out by Primas (1980a). To obtain such a general theory, one has to introduce a formulation comprising classical and quantum dynamics in one unified scheme. The description in terms of W^* algebras of bounded observables provides such a unified formulation. A general outline of this approach in the nonrelativistic case, is given by Primas (1980a, to be published). We will refer to these reviews for the necessary background, and for additional references.

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Our aim here is to give a description of a relativistic particle in terms of its W^* algebra $\mathcal{Q}$, in the (reducible) standard representation [in the sense of Haagerup (1975)]. As a fundamental application, we obtain the classical ($\hbar \to 0$) limit of the corresponding quantum system.

2. THE W^* ALGEBRA OF THE CLASSICAL PARTICLE

The phase space of a relativistic particle of kinematical mass m, capable of entering into interactions, is (Aaberge, 1977)

$$\Gamma = \left\{ \omega = (p^\mu, q^\mu) | (p^0 + mc)^2 - \mathbf{p}^2 > 0; p^0 > -mc \right\}$$
$$= M \times \mathbb{R}^4 \tag{1}$$

with the canonical symplectic form α

$$\alpha(p_\mu, q^\mu) = dp_\mu \wedge dq^\mu \tag{2}$$

The fundamental observables are the four-momentum p^μ, the four-"position" q^μ, the mass defect

$$\Delta m = \frac{1}{c}\left[(p^0 + mc)^2 - \mathbf{p}^2\right]^{1/2} - m. \tag{3}$$

and the Newton–Wigner type of position

$$\mathbf{x} = \mathbf{q} - \frac{\mathbf{p}}{p^0 + mc} q^0 \tag{4}$$

Observe that the two kinds of position observables are equal in the stationary system corresponding to the momentary rest frame of the system. We assume that $\mathbf{x}$ is the physical position. In the quantum case, $\mathbf{x}$ defines a self-adjoint operator, whereas q^μ does not.

The Poincaré group $\{\Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu, a^\mu\}$ is represented on p^μ, q^μ by the transformations

$$p^\mu \mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu p^\nu + m v^\mu(\mathbf{u})$$
$$q^\mu \mapsto \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu q^\nu + t v^\mu(\mathbf{u}) - a^\mu$$
$$v^\mu(\mathbf{u}) = [c(\gamma - 1), \mathbf{u}], \qquad \gamma(\mathbf{u}) = (1 - u^2/c^2)^{-1/2} \tag{5}$$

t is the invariant time.

The set of bounded observables of the system can be identified with the Lebesgue space $L^\infty(\Gamma)$ (Primas, 1980a). $L^\infty(\Gamma)$ is an Abelian W^* algebra, which we will call $\mathfrak{A}^{c\prime}$.

The nonlinearity of the phase space Γ leads to undesired properties of the coordinates. For instance, q_μ do not define self-adjoint operators by canonical quantization

$$q_\mu \mapsto i\hbar \frac{\partial}{\partial p^\mu} \tag{6}$$

We therefore introduce a new coordinatization, in which the phase space becomes linear. The quantum W^* algebra, then, can later on be defined in terms of the linear coordinates.

Consider the functions

$$\begin{aligned}
\eta^0 &= \tfrac{1}{2}\ln\left[\left(p^0+mc\right)^2-\mathbf{p}^2\right]-\ln mc \\
\eta^i &= p^i \\
\xi_0 &= \frac{mc\left[\left(p^0+mc\right)^2-\mathbf{p}^2\right]}{p^0+mc}\ln\left\{\frac{\left[\left(p^0+mc\right)^2-\mathbf{p}^2\right]^{1/2}}{mc}\right\}q_0 \\
\xi_i &= q_i - \frac{p_i}{p^0+mc}q^0
\end{aligned} \tag{7}$$

defining a map $\phi\colon \Gamma \to \mathbb{R}^8$ with the inverse map

$$\begin{aligned}
p^0 &= \left(\boldsymbol{\eta}^2 + m^2c^2e^{2\eta^0}\right)^{1/2} - mc \\
p^i &= \eta^i \\
q_0 &= \frac{\left(\boldsymbol{\eta}^2 + m^2c^2e^{2\eta^0}\right)^{1/2}}{m^2c^2e^{2\eta^0}\eta^0}\xi_0 \\
q_i &= \xi_i + \frac{\eta_i}{m^2c^2\eta^0e^{2\eta^0}}\xi^0
\end{aligned} \tag{8}$$

The functions ξ_μ are defined by

$$\xi_\mu(p^\mu, q^\mu) = \frac{\partial p_\nu}{\partial \eta^\mu}\left[\eta^\mu(p^\mu)\right]q^\nu \tag{9}$$

Hence,

$$\xi_\mu \, d\eta^\mu = q_\mu \, dp^\mu \tag{10}$$

We can conclude, that the map ϕ is a symplectomorphism with respect to the symplectic form

$$\alpha(p_\mu, q^\mu) = dp_\mu \wedge dq^\mu \tag{11}$$

It follows that the coordinates (η_μ, ξ^μ) are canonical coordinates for the Hamiltonian system. For a given Hamiltonian $H'(p_\mu, q^\mu)$, there correspond the Hamiltonian $H = H' \circ \phi$. Moreover, the map ϕ defines a *-isomorphism

$$\begin{aligned} \psi\colon L^\infty(\Gamma) &\to L^\infty(\mathbb{R}^8) \\ f' &\mapsto f = f' \circ \phi^{-1} \end{aligned} \tag{12}$$

defining another concrete representation of the W^* algebra $\mathfrak{A}^c$. We will call this representation $\tilde{\mathfrak{A}}^c$.

We have by this construction obtained a simplified description of $\mathfrak{A}^c$ in terms of canonical coordinates on a linear manifold, at the expense of the simplicity of the Hamiltonian functional form. However unsuitable this would be with regard to calculations, it is of minor importance in the study of asymptotic limits of the general structure. The implementation of the approach advocated by Primas in the nonrelativistic case (Primas, 1980b, 1980c) now becomes possible.

Before turning to the quantum particle, we introduce the last step in the construction of a formulation compatible with the quantum case. We represent $\mathfrak{A}^c$ as an Abelian operator algebra on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^8)$. This can be done (Primas, 1980a) by the identification of $\tilde{\mathfrak{A}}^c$ with the set of multiplication operators $\hat{\mathfrak{A}}^c$, defined by

$$\begin{aligned} \tilde{\mathfrak{A}}^c &\to \hat{\mathfrak{A}}^c \\ f(\omega) &\mapsto \hat{f} \in \mathfrak{B}(\mathcal{H}) \\ \hat{f}\psi(\omega) &= f(\omega)\psi(\omega) \end{aligned} \tag{13}$$

3. THE W^* ALGEBRA OF THE QUANTAL PARTICLE

An irreducible representation of the W^* algebra $\mathfrak{A}^q$ of the quantum particle, equivalent to the ones given by Aaberge (1978), is given by $\mathfrak{B}(L^2(\mathbb{R}^4))$, the set of bounded operators on the Hilbert space defined on the momentumlike space $\{\eta_\mu\} = \mathbb{R}^4$. In this representation, the four pairs (η_μ, ξ^μ) are Schrödinger pairs:

$$[\xi^\mu, \eta_\mu] = i\hbar\delta^\mu{}_\nu \tag{14}$$

Moreover,

$$\mathcal{B}(L^2(\mathbb{R}^4)) = \{\xi^\mu, \eta_\mu\}''$$

In general, if the quantum system is represented by four self-adjoint Schrödinger pairs (ξ^μ, η_μ) corresponding to canonical coordinates, and the W^* algebra is represented by the bounded operators generated by these operators, then

$$\hat{\mathcal{A}}^q = \{\xi^\mu, \eta_\mu\}'' \tag{15}$$

We now introduce the reducible standard representation $\hat{\mathcal{A}}^q \subset \mathcal{B}(\mathcal{H})$; $\mathcal{H} = L^2(\mathbb{R}^8)$, defined on the phase space $\mathbb{R}^8 = \{(\eta_\mu, \xi^\mu)\}$. The self-adjoint operators $\hat{\eta}_{\mu\hbar}, \hat{\xi}^\mu_\hbar$ are defined by the operator forms

$$\hat{\xi}^\mu_\hbar \psi(\eta_\mu, \xi^\mu) = \left(\xi^\mu + \frac{1}{2} i\hbar \frac{\partial}{\partial \eta_\mu}\right) \psi(\eta_\mu, \xi^\mu)$$

$$\hat{\eta}_{\mu\hbar} \psi(\eta_\mu, \xi^\mu) = \left(\eta_\mu - \frac{1}{2} i\hbar \frac{\partial}{\partial \xi^\mu}\right) \psi(\eta_\mu, \xi^\mu) \tag{16}$$

In this way we obtain a representation of the commutation relations (14), defining a von Neumann algebra

$$\hat{\mathcal{A}}^q_\hbar = \{\eta_{\mu\hbar}, \xi^\mu_\hbar\}'' \tag{17}$$

The "physical" operators $p^\mu \mapsto \hat{p}^\mu_\hbar$ and the bounded functions of $q^\mu, f(q^\mu) \mapsto \hat{f}_\hbar(q^\mu(\hat{\xi}^\mu_\hbar, \hat{\eta}^\mu_\hbar))$ are obtained via the spectral calculus and the Jordan product between noncommuting operators.

4. THE CLASSICAL LIMIT.

As a simple, although fundamental, application, we now consider the limit $\hbar \to 0$. It is in fact easy to see from the definitions (16) that

$$\hat{\mathcal{A}}^q_\hbar \overset{\hbar \to 0}{\to} \hat{\mathcal{A}}^c$$

in the following, rigorous sense (Primas, 1980b, 1980c):

$$\operatorname*{s-lim}_{\hbar \to 0} \hat{\eta}_{\mu\hbar} = \hat{\eta}_{\mu}$$

$$\operatorname*{s-lim}_{\hbar \to 0} \hat{\xi}^{\mu}_{\hbar} = \hat{\xi}^{\mu}$$

denoting strong limits in the resolvent sense, since the operators are unbounded.

The application of the formulation presented here to the Born–Oppenheimer type approximation, is the subject of a forthcoming paper (Grelland, to be published).

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